I. Envelopes and Evolutes

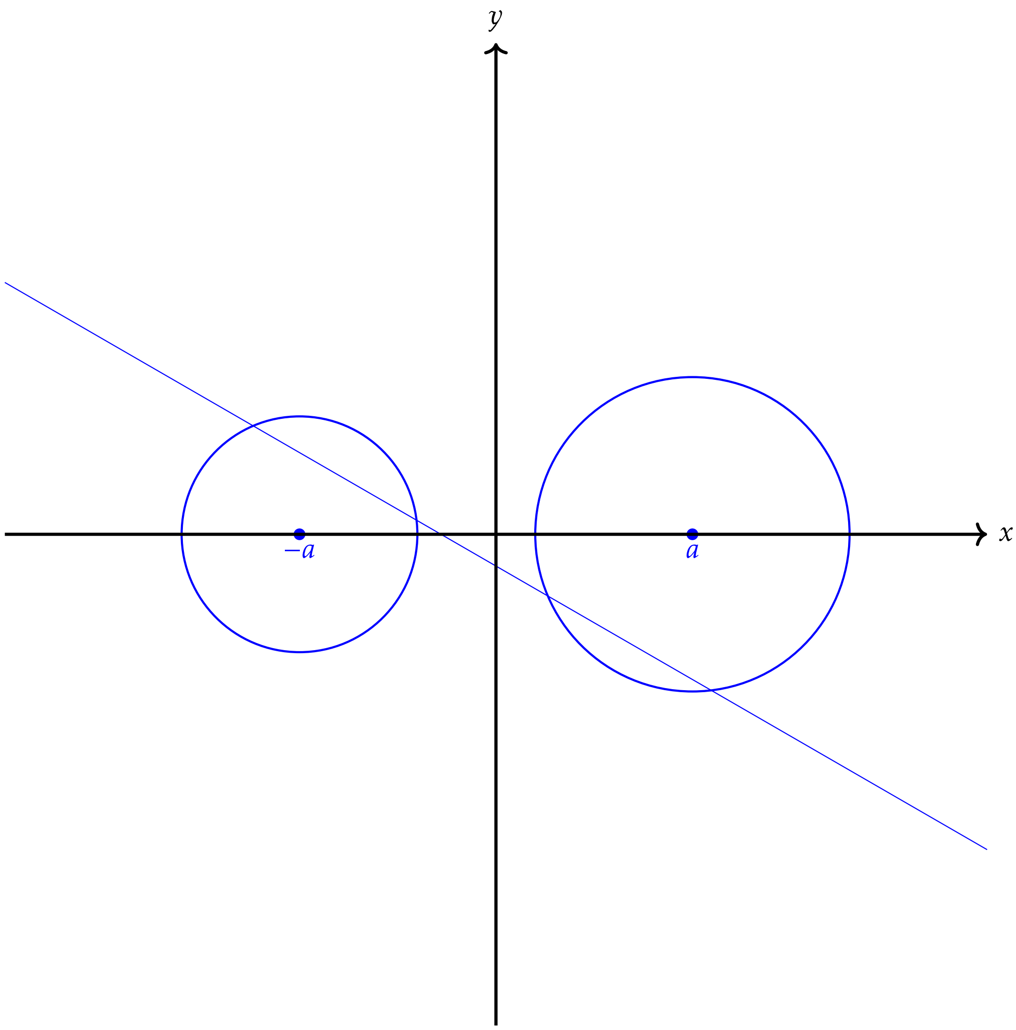
1. Sketch the family of curves
$$(x-a)^2 - y^2 + y^3 = 0,$$
where a is a parameter. Show that the usual method of finding an envelope, by eliminating a between the equations $F(x,y,a) = 0$ and $\partial F/\partial a = 0$, yields in this case not only the envelope but also an extraneous locus. How do you account for this?

   Prove that for a family of straight lines of the form $y = ax + P(a)$, where P is a polynomial of degree at least two, the standard method always yields a true envelope (that is, a curve whose slope at any point is the same as that of some straight line of the family which passes through that point).

   Find, in the form $\phi(x,y) = 0$, the envelope of the family of lines $y = ax + 3x - a^3$, and sketch it.

2. A family of plane curves has the property that if the tangent to $f(x,y)$ of any one of the curves intersects the x-axis in N, then the distance ON is equal to ky^2, where O is the origin and k is a positive constant. Find the equation of the particular curve of the family that passes through the point $(0,1)$ and sketch it.

3. Find the relation between p and α in order that the straight line
$$x\cos\alpha + y\sin\alpha = p$$
should cut the circles
$$(x-a)^2 + y^2 = b^2, \quad (x+a)^2 + y^2 = c^2,$$
in chords of equal length. Prove that the envelope of the lines satisfying this condition is a parabola, and find its equation. *CCE 1951 Paper 4 Q10*



We can see the length of the chords are $2\sqrt{b^2 - d_1^2}$ and $2\sqrt{c^2 - d_2^2}$ where d_1 and d_2 are the perpendicular distances to the lines. For these to be equal we must have $b^2 - d_1^2 = c^2 - d_2^2$.

$$\begin{aligned} & d_1 = |a\cos\alpha - p| \\ & d_2 = |-a\cos\alpha - p| = |a\cos\alpha + p| \\ \Rightarrow \quad & b^2 - (a\cos\alpha - p)^2 = c^2 - (a\cos\alpha + p)^2 \\ \Leftrightarrow \quad & b^2 + 2ap\cos\alpha = c^2 - 2ap\cos\alpha \\ \Leftrightarrow \quad & c^2 - b^2 = 4ap\cos\alpha \end{aligned}$$

Therefore our lines are: $x\cos\alpha + y\sin\alpha = \frac{c^2-b^2}{4a\cos\alpha}$, dividing by $\cos\alpha$ we find:

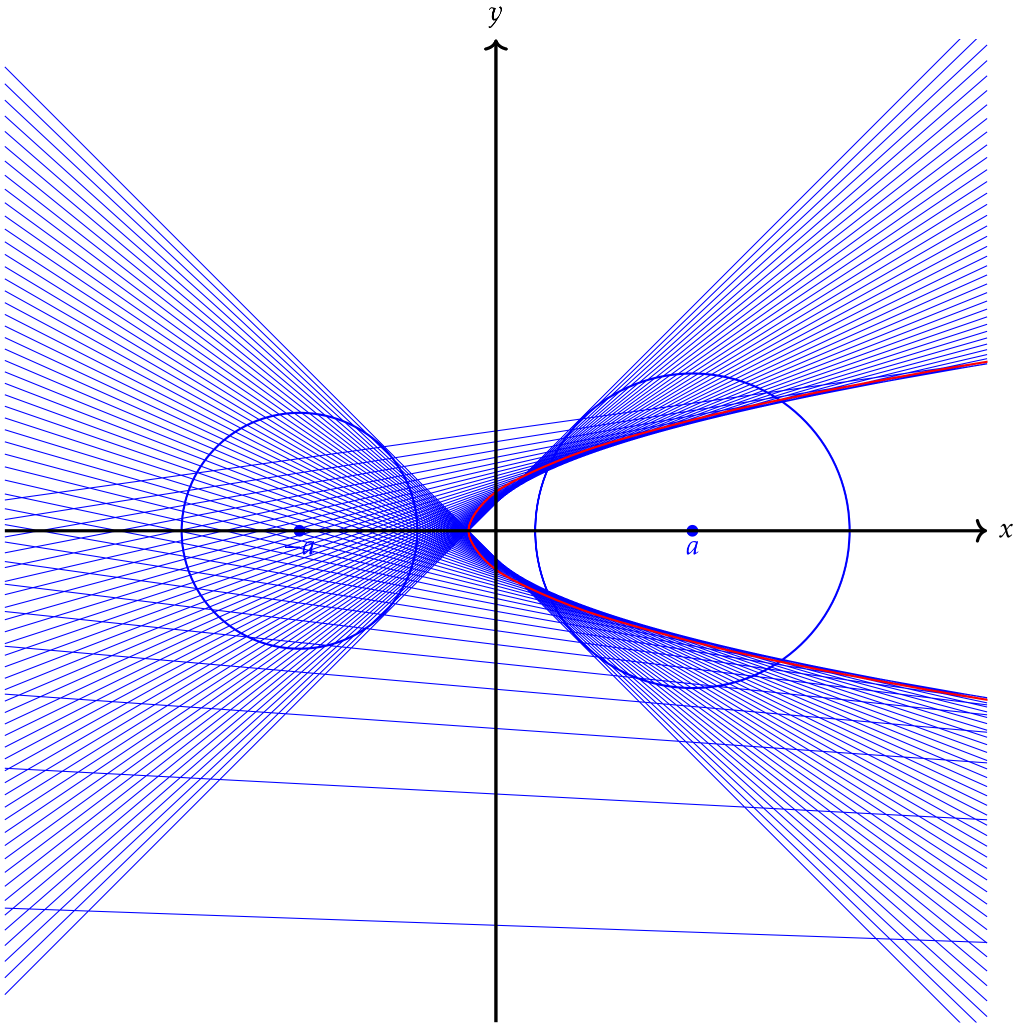
$$\begin{aligned} x + y\tan\alpha &= \frac{c^2 - b^2}{4a}\sec^2\alpha \\ &= \frac{c^2 - b^2}{4a}(1 + \tan^2\alpha) \\ \Rightarrow \quad 0 &= \left(\frac{c^2 - b^2}{4a}\right)\tan^2\alpha - y\tan\alpha + \left(\frac{c^2 - b^2}{4a} - x\right) \end{aligned}$$

We can find the limiting points as when the discriminant is zero, ie

$$0 = \Delta = y^2 - 4\left(\frac{c^2-b^2}{4a}\right)\left(\frac{c^2-b^2}{4a} - x\right)$$

$$\Rightarrow \qquad y^2 = 4\left(\frac{c^2-b^2}{4a}\right)\left(\frac{c^2-b^2}{4a} - x\right)$$

Which is clearly a parabola.



4. Prove that the envelope of the line $L\cos\theta + M\sin\theta = N$, where θ is a parameter, is the conic $L^2 + M^2 = N^2$. Two straight lines are drawn at right angles to each other, one touching $(x-a)^2 + y^2 = b^2$ and the other touching $(x+a)^2 + y^2 = c^2$. Prove that the envelope of the line joining their points of contact is the conic
$$(b^2+c^2)x^2 + (b^2+c^2-4a^2)y^2 + 2a(b^2-c^2)x + a^2b^2 + a^2c^2 - b^2c^2 = 0,$$
and that the tangents to this conic from any point on either circle meet the other circle at the ends of a diameter. *CCE 1930 Paper 4 Q1*

5. If the coordinates of a point in a curve are known functions of a single parameter t, find the equations of the tangent and normal at a given point in terms of the parameter. Prove that the equation of the normal to the curve
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$
may be written in the form $x\sin\phi - y\cos\phi + a\cos 2\phi = 0$, and find the equation of the envelope of the normal. *CCE 1926 Paper 2 Q10*

6. A particle is projected in a given vertical plane from a point O, the horizontal and vertical components of velocity being u and v respectively. If u, v are connected by a relation

$$au^2 + v^2 = 2gh,$$

where a, h are positive constants, shew that the envelope of the trajectories is a parabola whose vertex is at a height h above O and whose latus rectum is $4h/a$. Shew also that to reach the point Q on the envelope the elevation of the direction of projection must be $\tan^{-1}(2h/x)$, where x denotes the horizontal projection of OQ. A gun is mounted on a truck and can fire a shell of mass m in a vertical plane parallel to the rails, the mass of the gun and truck together being M. Find the envelope of the trajectories (i) on the assumption that the velocity of the shell relative to the gun is constant and equal to $\sqrt{2gh}$, (ii) on the assumption that the total kinetic energy immediately after the shell leaves the gun is constant and equal to mgh.

CCE 1931 Paper 4 Q9

7. Give a general account of the motion of a projectile, neglecting air resistance. Consider the possible paths through a given point P when the velocity at the point of projection O is given in magnitude, and the envelope of the paths when the direction is varied for a given magnitude. A fort is on the edge of a cliff of height h. Show that there is an annular region in which the fort is out of range of the ship, but the ship is not out of range of the fort, of area $8\pi kh$, where $\sqrt{2gk}$ is the velocity of the shells used by both. *CCE 1925 Paper 4 Q8*

8. Find the equations of the tangent and normal at the point $(at^2, 2at)$ of the parabola $y^2 = 4ax$. The normals at P, Q, the extremities of a focal chord of this parabola, meet the parabola again in $P'Q'$. Prove that the envelope of the chord $P'Q'$ is the parabola $y^2 = 32a(9a - x)$.

CCE 1922 Paper 1 Q8

9. Prove that the envelope of all parabolas of which the focus is at the origin and the vertex is on the circle $x^2 + y^2 = 2ax$ is the straight line $x = 2a$. *CCE 1919 Paper 2 Q11*

10. Circles are drawn with their centres on the circle $x^2 + y^2 = 1$ and touching the axis of y. Shew that the axis of y and the curve

$$4(x^2 + y^2 - 1)^2 = 27x^2$$

form the envelope of the system of circles; and trace the curve. *CCE 1914 Paper 1 Q14*

II. Sketching Families of Curves (Parameter Variation)

11. Sketch the curve $x^4 + y^4 - 2x^2a = 0$ for the values $2, 1, \frac{1}{4}, 0, -1$ of the parameter a.

12. Sketch the three curves

$$xy^2 = (a - x)^2(1 - x)$$

for the following three values of the parameter a:

$$a = \frac{1}{2}, 1, 2.$$

13. Sketch the curve $y = 3x^5 - 5ax^3$ for positive and negative values of the real number a, and hence determine the number and signs of the real roots of the equation

$$3x^5 - 5ax^3 + b = 0,$$

where b is real, in the various cases that may arise.

14. Sketch the cubic curve
$$(xy-12)(x+y-9)=a$$
   (i) for a small positive value of the constant a, and
   (ii) for a small negative value of the constant a.

   For what value of a does the curve have an isolated point?

15. Let
$$f(x)=k\cos x-\cos 2x,$$
where k is a constant, $k>0$. By considering the sign of $f'(x)$, or otherwise, find the greatest and least values taken by $f(x)$ for $0\le x\le \frac{1}{2}\pi$, distinguishing the various cases that arise according to the value of k. Sketch the graph of $y=f(x)$ for $0\le x\le\frac{1}{2}\pi$ in each case.

16. Sketch the curve $x^2=(y-k)^2(y-2k)$, where x, y are real variables and k is constant, in the three cases (i) $k<0$, (ii) $k=0$, (iii) $k>0$. Describe the nature of the singularity in each case.

17. Sketch the curves
$$x^n+y^n=1$$
for $n=-1,1,2,3,4$.

   Related: Sketch the curves $x^n+y^n=1$, for $n=10,11$, and $1/11$.

18. Prove that, if k is real and $|k|<1$, the function $\cot x+k\csc x$ takes all values as x varies through real values. Prove that, if $|k|>1$, the function takes all values except those included in an interval of length $2\sqrt{(k^2-1)}$.

   Give rough sketches of the graph of
$$y=\cot x+k\csc x$$
for $-\pi<x<\pi$, in the cases (i) $0<k<1$, (ii) $k>1$.

19. Sketch the curves $\cosh x=\dfrac{y\cosh\alpha}{\sin y}$ for different values of the parameter α ($\alpha\ge 0$), and for values of y between $-\pi$ and π. Show that, on the curve of parameter α, the function
$$\sinh(x+iy)-(x+iy)\cosh\alpha$$
is purely real, and indicate its direction of increase along the curve.

20. A point moves in the plane so that its distances from a fixed point P and a fixed line l (not through P) are in the ratio λ to 1. Describe the locus of the point and draw a sketch of the loci obtained for varying λ, indicating the effect as λ increases and the locus for $\lambda=1$.

21. A plane curve is such that the tangent at any point P is inclined at an angle $(k+1)\theta$ to a fixed line Ox, where k is a positive constant and θ is the angle xOP. The greatest length of OP is a. Find a polar equation for the curve. Sketch the curves for the cases $k=2, k=\frac{1}{2}$.

22. Let
$$f(x)=x^p(1-x)^q,$$
where $p>1$, $q>1$. Sketch the graph of $f(x)$ for $0\le x\le 1$. Show by means of sketches how f behaves in this interval for other positive values of p and q, distinguishing between different ranges of values of p and q so as to indicate the different types of curve that may occur.

III. Differential Equations (Families of Solutions)

23. The function f satisfies $f(-y) = -f(y)$ and is defined as follows for $y \geq 0$:

$$f(y) = y \quad \text{if } 0 \leq y \leq 1,$$

$$f(y) = 2 - y \quad \text{if } 1 \leq y \leq 2,$$

$$f(y) = 0 \quad \text{if } y \geq 2.$$

Solve the differential equation $y'' + f(y) = 0$ with initial conditions $y(0) = 0$, $y'(0) = c$. Sketch the solutions corresponding to initial conditions $y(0) = 0$, $y'(0) = c$ for $c = 1$, $c = \frac{4}{3}$ and $c = \frac{8}{3}$.

24. Find the solution of $\frac{dy}{dx} = xy(y-2)$ such that $y(0) = y_0$. Sketch the forms of solution that arise for $y_0 > 0$.

25. A particle moves along a straight line Ox so that when it is at a distance x from O its acceleration is n^2x directed away from O. The particle is projected towards O with velocity nb from a point at a distance a from O. Discuss the motion and sketch the (v, x) graphs for the cases $b > a, b = a, b < a$.